We thus arrive at the result:

$$\binom{cn}{a}-\binom{a}{1}\binom{cn-c}{a}+\binom{a}{2}\binom{cn-2c}{a}-\cdots+(-1)^a\binom{cn-ca}{a}=c^a.$$

II. Solution by Norman Anning, Chilliwack, B. C.

Define operators E and Δ as follows:

$$E^hf(n)=f(n+h),\qquad \Delta f(n)=f(n+1)-f(n)=(E-1)f(n).$$

Then the given expression,

$$\binom{cn}{a}-\binom{a}{1}\binom{cn-c}{a}+\binom{a}{2}\binom{cn-2c}{a}-\cdots+(-1)^a\binom{cn-ca}{a}$$

$$=(1-E^{-1})^a\binom{cn}{a}=(\Delta E^{-1})^a\binom{cn}{a}=\Delta^aE^{-a}\binom{cn}{a}=\Delta^a\binom{cn-ca}{a}.$$

Now

$$\binom{cn-ca}{a}=\frac{1}{a!}\{c^an^a+\text{terms in } n \text{ of degree lower than } a\}.$$

Hence,

$$\Delta^a\binom{cn-ca}{a}=\frac{1}{a!}\{c^a\cdot a!+0\}=c^a.$$

446. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Solve the equations

$$x^2(y-z)=l^2(m-n),\qquad y^2(z-x)=m^2(n-l),\qquad z^2(x-y)=n^2(l-m).$$

Solution by H. S. Uhler, Yale University.

An equation involving only one variable x may be obtained in the following manner. Equating the expressions for z derived from the first two given equations we find

$$y-l^2(m-n)/x^2=x+m^2(n-l)/y^2$$

or

$$x^3y^2-x^2y^3+m^2(n-l)x^2+l^2(m-n)y^2=0. \tag{1}$$

Substituting the value of z from the first in the third given equation and reducing, we obtain

$$x^5y^2-x^4y^3-n^2(l-m)x^4-2l^2(m-n)x^3y+2l^2(m-n)x^2y^2+l^4(m-n)^2x-l^4(m-n)^2y=0. \tag{2}$$

Multiplying (1) by x^2 and subtracting (2) from the product gives, after removing the factor $m-n$,

$$(lm-mn+nl)x^4-2l^2x^3y+l^2x^2y^2+l^4(m-n)x-l^4(m-n)y=0. \tag{3}$$

Multiplying (1) by l^2, (3) by y, adding, and dividing by x,

$$(lm-mn+nl)x^3y-l^2x^2y^2+l^2m^2(n-l)x+l^4(m-n)y=0. \tag{4}$$

Adding (3) and (4) and dividing by x,

$$(lm-mn+nl)x^3+(lm-mn+nl-2l^2)x^2y+l^2[l^2(m-n)+m^2(n-l)]=0. \tag{5}$$

Substituting the expression for y from (5) in (4) and removing the factors $(l-m)(l-n)$ we derive the following quadratic in x^3, namely

$$(lm-mn+nl)^2x^6-2l^4[m^2(l-n)+n^2(l-m)]x^3-l^6(mn-nl+lm)(nl-lm+mn)=0.$$

Consequently

$$x^3=l^3$$

and

$$x^3=\frac{l^3[l^2(m-n)^2-m^2n^2]}{(lm-mn+nl)^2}.$$

Finally, the six values of x are

$$l,\ l\omega,\ l\omega^2,\ \frac{l[l^2(m-n)^2-m^2n^2]^{\frac{1}{3}}}{(lm-mn+nl)^{\frac{2}{3}}},\ \frac{l[l^2(m-n)^2-m^2n^2]^{\frac{1}{3}}\omega}{(lm-mn+nl)^{\frac{2}{3}}},\ \frac{l[l^2(m-n)^2-m^2n^2]^{\frac{1}{3}}\omega^2}{(lm-mn+nl)^{\frac{2}{3}}},$$

where

$$\omega \equiv (-1+i\sqrt{3})/2.$$

The corresponding values of y and z may be obtained at once by cyclical permutation of the letters l, m, and n.

Remarks: The above analysis was verified by direct substitution of the six sets of values in the given equations. The elimination of y between equations (1) and (2) can be performed in a more elegant manner by Sylvester's dialytic method, but the simplification of the determinant of the eighth order requires too much space for publication.

Also solved by A. H. Holmes and Norman Anning.

GEOMETRY.

471. Proposed by C. N. SCHMALL, New York City.

In the ellipse $x^2/a^2 + y^2/b^2 = 1$, an equilateral hexagon is inscribed with two sides parallel to the major axis. In the major auxiliary circle the same thing is done. If H_1 and H_2 be the sides of the hexagons, and e the eccentricity of the ellipse, show that $H_1 : H_2 :: 4 - 2e^2 : 4 - e^2$.

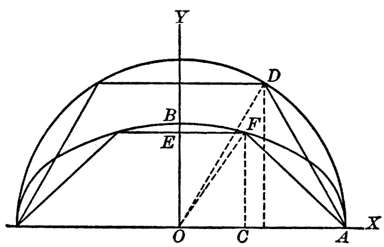
Solution by J. A. Caparo, University of Notre Dame.

Since AD is the side of a regular hexagon inscribed in the major circle of the ellipse $x^2/a^2 + y^2/b^2 = 1$, we have

$$DA = OA = a = H_2$$

and since

$$e^2 = \frac{a^2-b^2}{a^2}, \qquad b^2 = H_2^2(1-e^2), \qquad \therefore\ \frac{x^2}{H_2^2} + \frac{y^2}{H_2^2(1-e^2)} = 1. \tag{1}$$



Now $EF = H_1/2$; therefore, from (1) we have

$$FC^2 = y^2 = \frac{(4H_2^2 - H_1^2)(1-e^2)}{4}.$$

Also $CA = OA - OC$. Hence, $CA = H_2 - (H_1/2)$; and since $FA^2 = FC^2 + CA^2$, we have

$$4H_1^2 = (4H_2^2 - H_1^2)(1-e^2) + (2H_2 - H_1)^2.$$

Write $H_1/H_2 = x$. Then, substituting and reducing, we have

$$x^2(4-e^2) + 4x + 4(e^2-2) = 0.$$

Solving for x, we have

$$x = \frac{-2 \pm 2(e^2-3)}{4-e^2}.$$

Using the negative sign, we have $x = (4-2e^2)/(4-e^2)$; and since $x = H_1/H_2$,

$$H_1 : H_2 :: 4 - 2e^2 : 4 - e^2.$$

Also solved by C. E. Horne, Frank Irwin, Harold T. Davis, Norman Anning, Elijah Swift, and C. N. Schmall.